MR3318369 14K10 14G35 14J15
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Polarized real tori. (English summary)
J. Korean Math. Soc. **52** (2015), *no. 2,* 269–331.

For a given integer g, let $\mathbb{H}_g = \{\Omega \in \mathbb{C}^{(g,g)} : \Omega = {}^t\Omega,\ \mathrm{Im}\,\Omega > 0\}$ be the Siegel upper half-plane of degree g, and let $\mathrm{Sp}(g,\mathbb{R}) = \{M \in \mathbb{R}^{(2g,2g)} : {}^tMJ_gM = J_g\}$ be the symplectic group of degree g. $F^{k,l}$ is the set of $k\times l$-matrices with entries in a commutative ring F and ${}^tM$ denotes the transpose of M, $J_g = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$.

$\mathrm{Sp}(g,\mathbb{R})$ acts transitively on $\mathbb{H}_g$ by

$$M\cdot\Omega = (A\Omega+B)(C\Omega+B)(C\Omega+D)^{-1},$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(g,\mathbb{R})$ and $\Omega\in\mathbb{H}_g$. Letting

$$\Gamma_g = \mathrm{Sp}(g,\mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(g,\mathbb{R}) \colon A,B,C,D \text{ integral} \right\}$$

be the Siegel upper modular group of degree g, this group acts properly discontinuously on $\mathbb{H}_g$. Siegel investigated the geometry of $\mathbb{H}_g$ and the automorphic forms on it, and found a fundamental domain $\mathcal{F}_g$ for $\Gamma_g\backslash\mathbb{H}_g$ for which he calculated the volume. $\mathcal{A}_g = \Gamma_g\backslash\mathcal{H}_g$ is called the Siegel modular variety and is an important arithmetic variety because it is the moduli space of principally polarized abelian varieties of dimension g. Satake gave a canonical compactification of $\mathcal{A}_g$, and Baily proved that the Satake compactification of $\mathcal{A}_g$ is a normal projective variety. Then this was generalized to bounded symmetric varieties and so a theory of smooth compactifications of bounded symmetric domains was developed by the GIT theory. Faltings and Chai studied the moduli of abelian varieties over the integers and gave the analogous result of the Eichler-Shimura theorem describing Siegel modular forms in terms of cohomology of local systems on $\mathcal{A}_g$.

Let $\mathcal{P}_g = \{Y \in \mathbb{R}^{(g,g)} : Y = {}^tY > 0\}$ be an open convex cone in $\mathbb{R}^N$, $N = g(g+1)/2$. $\mathrm{GL}(g,\mathbb{Z})$ acts transitively on $\mathbb{P}_g$ by $A\circ Y = AY{}^tA$, and this action is naturally induced from the symplectic action. Thus $\mathcal{P}_g$ is a symmetric space diffeomorphic to $\mathrm{GL}(g,\mathbb{R})/\mathrm{O}(g)$. Considering the arithmetic subgroup $\mathrm{GL}(g,\mathbb{Z}) \subseteq \mathrm{GL}(g,\mathbb{R})$ and using reduction theory, one has the Minkowski fundamental domain $\mathfrak{R}_g$ for the action of $\mathrm{GL}(g,\mathbb{Z})$ on $\mathcal{P}_g$.

The main aim of this article is to give the arithmetic-geometric meaning of the Minkowski fundamental domain $\mathfrak{R}_g$. The author starts by introducing a notion of polarized real abelian varieties, relating special real tori to polarized real abelian varieties. The author proves that $\mathcal{P}_g$ parametrizes principally polarised real tori of dimension g and that the Minkowski modular space $\mathfrak{J}_g = \mathrm{GL}(g,\mathbb{Z})\backslash\mathcal{P}_g$ can be regarded as a moduli space for principally polarized real tori of dimension g. Also, the author studies smooth line bundles over polarized real tori by relating them to holomorphic line bundles over the associated polarized abelian variety.

If $G^M = \mathrm{GL}(g,\mathbb{R}) \ltimes \mathbb{R}^g$ denotes the semidirect product with multiplication $(A,a)\cdot(B,b) = (AB, a{}^tB^{-1}+b)$, then there is a natural action of G^M on the Minkowski Euclid space $\mathcal{P}_g\times\mathbb{R}^g$. The discrete subgroup $G^M(\mathbb{Z}) \subset G^M$ acts on $\mathbb{P}_g\times\mathbb{R}^g$ properly discontinuously. Associating a principally polarized real torus of dimension g to each equivalence class in $\mathfrak{J}_g$, the quotient space $G^M(\mathbb{Z})\backslash(\mathcal{P}_g\times\mathbb{R}^g)$ can be regarded as a family of principally polarised real tori of dimension g. For two elements $[Y_1]\neq[Y_2]\in\mathcal{P}_g$, $\Lambda_i =$

$Y_i\mathbb{Z}^g$, and $T_i = \mathbb{R}^g/\Lambda_i$, T_1 and T_2 are diffeomorphic as smooth manifolds, but not as polarized real tori.

There are three essential properties of the Siegel modular variety $\mathcal{A}_g$: It is the moduli space of principally polarized abelian varieties of dimension g, it has the structure of a quasi-projective complex algebraic variety which is defined over $\mathbb{Q}$, and it has the canonical compactification (Satake-Baily-Borel compactification defined over $\mathbb{Q}$).

Silhol constructed a topologically ramified covering of $\mathfrak{I}_g$, and he gave a compactification of the underlying moduli space. Neither the moduli space nor the compactification has an algebraic structure. By considering real abelian varieties with suitable level structure, Goresky and Tai showed that the moduli space of real principally polarized abelian varieties with level $4m$-structure coincides with the set of real points of a quasi-projective algebraic variety defined over $\mathbb{Q}$ and consists of finitely many copies of $\mathfrak{G}_g(4m)\backslash\mathcal{P}_g$ with discrete subgroup $\mathfrak{G}_g(4m)$ of $\mathrm{GL}(g,\mathbb{Z})$, $\mathfrak{G}_g(4m) = \{\gamma \in \mathrm{GL}(g,\mathbb{Z})\colon \gamma \equiv I_g \pmod{4m}\}$.

The author gives the basic properties of the symplectic group $\mathrm{Sp}(g,\mathbb{R})$, of real abelian varieties and the moduli of these. A thorough treatment of the compactifications of the moduli space of real abelian varieties is given. Then the concept of polarized real tori is considered, their properties are studied, and several examples are given. The study of smooth line bundles over real tori leads to the study of polarized real tori by relating the smooth line bundles to holomorphic line bundles over the associated complex tori, and this also works the other way. Using this, a real torus can be embedded in a complex projective space and thus smoothly into a real projective space.

The author gives the concept of holomorphic line bundles on complex tori, and studies the moduli space of polarized real tori. Basic geometric properties of the Minkowski fundamental domain $\mathfrak{R}_g$ are given. It is proved that $\mathcal{P}_g$ parametrises principally polarised real tori of dimension g and that $\mathfrak{I}_g$ is their moduli space with universal family $G^M(\mathbb{Z})\backslash(\mathcal{P}_g \times \mathbb{R}^g)$.

The article ends by presenting some problems related to real polarized tori suited for further investigation.

This is a nice article, but it is not at all self-contained, and it demands advanced knowledge of abelian varieties and modular theory. It illustrates a lot of new and old techniques, and is of value in the field. *Arvid Siqveland*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.